ELSEVIER

Journal of Geometry and Physics 44 (2003) 523–538

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

A local approach to dimensional reduction
I. General formalism

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Received 28 May 2001; received in revised form 4 March 2002

Abstract

We present a formalism for dimensional reduction based on the local properties of invariant cross-sections ("fields") and differential operators. This formalism does not need an ansatz for the invariant fields and is convenient when the reducing group is non-compact.

In the approach presented here, splittings of some exact sequences of vector bundles play a key role. In the case of invariant fields and differential operators, the invariance property leads to an explicit splitting of the corresponding sequences, i.e. to the reduced field/operator. There are also situations when the splittings do not come from invariance with respect to a group action but from some other conditions, which leads to a "non-canonical" reduction.

In a special case, studied in detail in the second part of this article, this method provides an algorithm for construction of conformally invariant fields and differential operators in Minkowski space.

MSC: 37J15; 58Z05; 53Z05; 78A25; 81T40

Subj. Class.: Differential geometry; Classical field theory

Keywords: Differential geometry; Group action and symmetry; Invariant fields; Invariant differential operators; Non-compact dimensional reduction

1. Introduction

Symmetry and reduction have always—sometimes implicitly—been among the main tools in the arsenal of theoretical physics. Many methods that utilize the symmetry of physical systems have been proposed.

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PII: S0393-0440(02)00058-X

In this paper, we develop a technique for dimensional reduction of invariant vector fields and one-forms, as well as for reduction of invariant differential operators. We use the methods of differential geometry which allow us to make the reduction procedure simple and natural. In the second part of the paper [5] (briefly referred to as Part II), we apply our method to the problem of obtaining conformally invariant fields and differential operators in Minkowski space.

The main ideas of our method are the following. Let the Lie group G act by bundle morphisms on the vector bundle ξ over a smooth finite-dimensional manifold B, and let $C^\infty(\xi)_G$ stand for the set of all G-invariant sections of ξ. Our goal is to construct the *reduced bundle* ξ_G, i.e. a bundle the space of whose sections, $C^\infty(\xi_G)$, is in a bijective correspondence with $C^\infty(\xi_G)$. We give the construction of the reduced bundle ξ_G (if certain conditions are satisfied).

The construction of the reduced bundle is easily applicable to the important particular cases of the tangent and cotangent bundles of B. In these two cases, the action of G naturally yields certain G-intertwining short exact sequences of vector bundles. It turns out that these sequences are very useful for the classification of the G-invariant vector fields and one-forms.

The dimensional reduction of differential operators is based on the jet bundle picture. The language of jet bundles reveals the geometry of the differential operators and—on the technical side—reduces the operations on differential operators to simple algebraic manipulations. The essential ingredient of the dimensional reduction of a differential operator in this formalism is the restriction of the operator to a submanifold of B. To perform this restriction, one has to find splittings of certain short exact sequences of jet bundles. In the case of G-invariant differential operators, this splitting is provided automatically by the G-invariance.

Our method is purely geometric, using only the geometrically natural structures on the manifold. It applies to any smooth finite-dimensional manifold B and to any Lie group G. We note that the method we propose can be applied whether or not the group G is compact or not. If G is non-compact, some interesting phenomena may occur—a simple example is given in Appendix A of Part II.

This method is applicable to tensor products of vector bundles and to spinor bundles, which we plan to study in near future. We believe that our technique can be useful in Kaluza–Klein-type theories.

The plan of the first part of the paper is the following. In Section 2 we describe the method for dimensional reduction, paying special attention to the case of dimensional reduction of the tangent and cotangent bundles. In Section 3 we briefly introduce some notions from the theory of differential operators on vector bundles, and in Sections 4 and 5 we develop the technique for dimensional reduction of invariant differential operators. In Section 6 we explain how to reduce the action of a Lie group K whose action on ξ commutes with that of the reducing group G.

In the second part of the paper, we apply our method to obtain conformally invariant vector fields, one-forms and differential operators in Minkowski space. This construction is based on the observation of Dirac from the 1930s that the conformal group in Minkowski space is locally isomorphic to the orthogonal group in the six-dimensional space with signature (2, 4). Our method allows us to find the global transformation laws of the fields, the invariant subbundles, and the so-called equations of conformal electrodynamics as well as the admissible gauge transformations.

2. Dimensional reduction of tangent and cotangent bundles

Here we describe the method of dimensional reduction, paying special attention to the dimensional reduction of the tangent and cotangent bundles of a smooth manifold. All manifolds, bundles and mappings in the text are supposed to be smooth (C^∞).

Let B be a finite-dimensional manifold and $\xi = (E, \pi, B)$ be a finite-dimensional vector bundle over B; let $\xi_b = \pi^{-1}(b)$ denote the fiber of ξ over the point $b \in B$. Let $C \subset B$ be a submanifold of B, and let ξ_C (or $\xi|_C$) stand for the bundle $i^*\xi$ induced by the natural embedding $i : C \hookrightarrow B$; in other words, $\xi_C = (E', \pi', C)$, where $E' := \pi^{-1}(C)$ and π' is the restriction of π to E'. By "dimension of the bundle" we will mean the dimension of the typical fiber. The vector space of all sections of ξ will be denoted by $C^\infty(\xi)$.

Let the Lie group G act from the left on ξ by vector bundle morphisms, and let $T : G \times E \to E$ and $t : G \times B \to B$ be the actions of G on the total space and the base, respectively (i.e. $\pi \circ T_g = t_g \circ \pi$ and the restriction $T_g : \xi_b \to \xi_{t_g(b)}$ be a linear isomorphism for each $g \in G$). This action naturally induces an action of G on $C^\infty(\xi)$ by

$$g(\psi) := T_g \circ \psi \circ t_g^{-1}, \quad g \in G, \quad \psi \in C^\infty(\xi).$$

The subspace of $C^\infty(\xi)$ consisting of all G-invariant sections will be denoted by $C^\infty(\xi)_G$.

Under certain conditions, there exists a vector bundle ξ_G, called a *reduced vector bundle*, such that the space $C^\infty(\xi_G)$ of all its sections is in a bijective correspondence

$$\theta : C^\infty(\xi_G) \to C^\infty(\xi)_G \tag{1}$$

with the space $C^\infty(\xi)_G$ of all G-invariant sections of ξ. The group G will be referred to as a *reducing group* and the procedure as a *G-dimensional reduction*. Below we discuss the conditions imposed on the action of G on ξ.

Condition A. The G-orbits on B are of one and the same type and form a locally trivial bundle

$$p : B \to B/G, \tag{2}$$

where p is the natural projection.

The fibers of (2) are homogeneous G-spaces of type G/H, where H is some closed subgroup of G. The group G acts naturally on G/H from the left by $g[g'] := [gg']$. The local triviality of (2) means that each point $x \in B/G$ has a neighborhood $V \subseteq B/G$ such that there exists an isomorphism $\Phi : p^{-1}(V) \to V \times (G/H)$ satisfying the relation

$$\Phi \circ t_g(b) = (p(b), g[\pi_2 \circ \Phi(b)]),$$

where $b \in p^{-1}(V)$, $g \in G$, and π_2 is the canonical projection

$$\pi_2 : V \times (G/H) \to G/H : (v, [g]) \mapsto [g].$$

Condition B imposes restrictions on the action T of G on the total space E. If G_b is the stationary group of $b \in B$, then the restriction $T : G_b \times \xi_b \to \xi_b$ determines a linear representation of G_b in ξ_b. Let $\mathrm{st}\,\xi_b$ be the subspace of ξ_b consisting of all vectors fixed with respect to the representation of G_b in it:

$$\mathrm{st}\,\xi_b := \{u \in \xi_b | T_g(u) = u \ \ \forall g \in G_b\}.$$

The importance of the subspaces $\mathrm{st}\,\xi_b$ in the construction of ξ_G is due to the fact that if ψ is a G-invariant section of ξ, then $\psi(b) \in \mathrm{st}\,\xi_b$ for each $b \in B$.

Condition B. The set of vector subspaces $\mathrm{st}\,\xi_b \subseteq \xi_b$ form a vector subbundle $\mathrm{st}\,\xi$ which will be called a *stationary subbundle* of ξ.

If the above two conditions are satisfied, the vector bundle ξ will be called a *G-reducible vector bundle*. Its reduced vector bundle, ξ_G, is a vector bundle over B/G and has the same dimension as $\mathrm{st}\,\xi$. The local coordinate realizations of $\mathrm{st}\,\xi$ can be constructed as follows. Let $\{V_\alpha\}$ be a cover of B/G, and $\phi_\alpha : V_\alpha \to B$ be local sections of the bundle (2). Let $N_\alpha := \phi_\alpha(V_\alpha)$ be the graphs of ϕ_α (N_α are submanifolds of B transversal to the orbits of G and of dimension equal to the one of B/G), and $i_\alpha : N_\alpha \hookrightarrow B$ be the natural embeddings. Then the restrictions $\xi_G|_{N_\alpha} := \mathrm{st}\,\xi|_{N_\alpha} = i_\alpha^*(\mathrm{st}\,\xi)$ are the coordinate realizations of ξ_G. The cocycle gluing these representations can be naturally constructed with the help of the action of G on ξ [4]. Since the general procedure contains many technicalities, and in the physical example considered in the second part of the paper we do not need more than one chart, we do not treat the general case.

Remark 2.1. In general, the reduced bundle ξ_G is constructed as the set of its coordinate realizations and the morphisms of transition between them. A coordinate realization of ξ_G is obtained by taking a submanifold $N \subset B$ transversal to the G-orbits in B (i.e. by choosing a section of the bundle (2) whose image is transversal to the G-orbits), and restricting the base of the stationary subbundle $\mathrm{st}\,\xi$ to N. The transition morphisms come from the action of the group G. It is possible that the bundle (2) does not have a global section. In this case we have to consider a sufficient set of transversal local sections. The maximal atlas of the reduced bundle ξ_G is the set of all transversal local sections of $p : B \to B/G$ and the corresponding transition morphisms.

A different choice of a local realization N of B/G would amount only to a reparametrization and would not change the essential features of the reduced objects. In this sense, we can say that θ and ξ_G do not depend on N.

Remark 2.2. The set $C^\infty(\xi)_G$ of all G-invariant sections of ξ form a module over the set of G-invariant functions in the base B because each G-invariant function is constant on each orbit of G in B. On the other hand, the sections, $C^\infty(\xi_G)$, of the reduced bundle ξ_G are a module over the ring $C^\infty(B/G)$. From our construction, it is clear that the map θ is a homomorphism of the $C^\infty(B/G)$-module $C^\infty(\xi_G)$ to the $C^\infty(B)_G$-module $C^\infty(\xi)_G$.

Let us consider in detail the case of the tangent bundle $\tau(B) = (T(B), \pi, B)$ of a finite-dimensional manifold B. Let the action of the Lie group G on B satisfy Condition A.

Then the tangent lift of t, $t_* : G \times T(B) \to T(B)$, defines an action (t_*, t) of G on $\tau(B)$ which turns $\tau(B)$ into a G-reducible vector bundle. Let $\tau^{\mathrm{v}}(B)$ be the *vertical subbundle* of $\tau(B)$ which by definition consists of the vectors tangent to the fibers of the bundle (2):

$$\tau^{\mathrm{v}}(B)_b := \tau(p^{-1}(p(b)))_b.$$

Let $p^*\tau(B/G)$ be the bundle over B induced from $\tau(B/G)$ by the projection $p : B \to B/G$; clearly, G acts trivially on $p^*\tau(B/G)$. Then it is known [2, Section IX.1] that there exists a natural short exact sequence (SES)

$$0 \to \tau^{\mathrm{v}}(B) \xrightarrow{i} \tau(B) \xrightarrow{j} p^*\tau(B/G) \to 0, \tag{3}$$

where i is the natural embedding and j is the natural projection. This SES is *G-intertwining*, i.e. the action of G commutes with the morphisms i and j. Below, we explain how to perform a G-dimensional reduction of (3); we consider only the case of the bundle (2) being globally trivial since it is sufficient for our purposes.

Let us choose a global section of (2) and denote its graph by N. Then the restrictions of $\operatorname{st}\tau^{\mathrm{v}}(B)$, $\operatorname{st}\tau(B)$ and $\operatorname{st} p^*\tau(B/G)$ to N are coordinate realizations of the reduced bundles $\tau^{\mathrm{v}}(B)_G$, $\tau(B)_G$ and $(p^*\tau(B/G))_G$, respectively. Obviously, the SES

$$0 \to \tau^{\mathrm{v}}(B)_b \xrightarrow{i} \tau(B)_b \xrightarrow{j} p^*\tau(B/G)_b \to 0 \tag{4}$$

of G_b-modules over $b \in N$ is (trivially) intertwining with respect to the corresponding representations of the stationary subgroup G_b.

Now we want to restrict the G_b-modules $\tau^{\mathrm{v}}(B)_b$, $\tau(B)_b$, and $p^*\tau(B/G)_b$ to their stationary submodules, $\operatorname{st}\tau^{\mathrm{v}}(B)_b$, $\operatorname{st}\tau(B)_b$, and $\operatorname{st} p^*\tau(B/G)_b \cong p^*\tau(B/G)_b$, respectively. Do the stationary submodules form a SES?

In general, the answer to the above question is negative. To understand why, let us consider the SES of G-modules

$$0 \to L_0 \xrightarrow{i} L \xrightarrow{j} L_1 \to 0$$

intertwining with respect to the corresponding representations D_0, D, D_1 of G, i.e. such that

$$i \circ D_0(g) = D(g) \circ i, \qquad j \circ D(g) = D_1(g) \circ j$$

for each $g \in G$. Then $i(L_0)$ is an invariant subspace of L. Each vector $l \in L$ can be represented as

$$l = \begin{pmatrix} i(l_0) \\ \tilde{l} \end{pmatrix} = \begin{pmatrix} i(l_0) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{l} \end{pmatrix}, \qquad \begin{pmatrix} i(l_0) \\ 0 \end{pmatrix} \in i(L_0)$$

for some $l_0 \in L_0$. Note that this representation of l is not unique since L_1 is not naturally embedded in L. The fact that $j \circ i = 0$ implies that $j(l) = j((0, \tilde{l})^{\mathrm{T}})$.

In this representation, the action of $D(g)$ can be written as

$$D(g)l = \begin{pmatrix} d_{00}(g) & d_{01}(g) \\ 0 & d_{11}(g) \end{pmatrix} \begin{pmatrix} i(l_0) \\ \tilde{l} \end{pmatrix} = \begin{pmatrix} d_{00}(g)i(l_0) + d_{01}(g)\tilde{l} \\ d_{11}(g)\tilde{l} \end{pmatrix},$$

and

$$D_1(g)\circ j(l) = D_1(g)\circ j\begin{pmatrix} i(l_0)\\ \tilde{l}\end{pmatrix} = j\circ D(g)\begin{pmatrix} i(l_0)\\ \tilde{l}\end{pmatrix}$$
$$= j\begin{pmatrix} d_{00}(g)i(l_0)+d_{01}(g)\tilde{l}\\ d_{11}(g)\tilde{l}\end{pmatrix} = j\begin{pmatrix} 0\\ d_{11}(g)\tilde{l}\end{pmatrix}.$$

We see that $j(l)\in \operatorname{st} L_1$ if and only if $\forall g\in G, d_{11}(g)\tilde{l}=\tilde{l}$, while $l\in \operatorname{st} L$ if and only if $\forall g\in G$,

$$d_{00}(g)i(l_0)+d_{01}(g)\tilde{l} = i(l_0), \qquad d_{11}(g)\tilde{l}=\tilde{l}.$$

Hence, in general, the inclusion $j(\operatorname{st} L)\subset \operatorname{st} L_1$ is strict, i.e. the map $j:\operatorname{st} L\to \operatorname{st} L_1$ may fail to be an epimorphism. If, however, the representation D is decomposable (i.e. if L is a direct sum of two invariant subspaces), then the sequence

$$0\to \operatorname{st} L_0 \xrightarrow{i} \operatorname{st} L \xrightarrow{j} \operatorname{st} L_1 \to 0$$

will be a SES of G-modules. The obstruction to decomposability can be studied by using tools of algebraic topology; we are planning to explain this in detail elsewhere.

If Condition A is satisfied, then the representation of G_b in $\tau(B)_b$ is decomposable. The proof of the decomposability goes as follows. Since $i(\tau^{\mathrm{v}}(B))$ is a G-invariant subbundle of $\tau(B), i(\tau^{\mathrm{v}}(B)_b)$ is a G_b-invariant subspace of $\tau(B)_b$. Condition A guarantees that for each $b\in B$, there exists a submanifold of B, namely

$$W^{(b)} := \{\Phi^{-1}(x, \pi_2\circ\Phi(b)) | x\in V\subseteq B/G\}$$

(V is an open subset of B/G containing $p(b)$), which is transversal to the orbits of G in B, contains b, and consists of points with one and the same stationary group G_b. This means that G_b acts trivially on $W^{(b)}$ and, hence, on $\tau(W^{(b)})$, so $\tau(W^{(b)})_b$ is the invariant complement of $i(\tau^{\mathrm{v}}(B)_b)$ with respect to the representation of G_b in $\tau(B)_b$. Thus, the representation of G_b in $\tau(B)_b$ is decomposable, which yields the SES

$$0\to \operatorname{st}\tau^{\mathrm{v}}(B)_b \xrightarrow{i} \operatorname{st}\tau(B)_b \xrightarrow{j} p^*\tau(B/G)_b \to 0.$$

Finally, taking into account that $(p^*\tau(B/G))_G\cong\tau(N)$, we obtain that after a G-dimensional reduction, (3) converts into the SES

$$0\to \tau^{\mathrm{v}}(B)_G \xrightarrow{i} \tau(B)_G \xrightarrow{j} \tau(N)\to 0. \tag{5}$$

The G-dimensional reduction of the cotangent bundle $\tau^*(B)$ is similar. The action t of G on B generates a natural contragradient action $(\hat{t}, t) := (t^{*-1}, t)$ of G on $\tau^*(B)$. The decomposability of the action (t_*, t) implies the decomposability of $(\hat{t}, t)$, hence after a G-dimensional reduction the dual of (3) G-intertwining SES goes into the SES

$$0\leftarrow (\tau^{\mathrm{v}}(B)^*)_G \xleftarrow{i^*} \tau^*(B)_G \xleftarrow{j^*} \tau^*(N)\leftarrow 0.$$

In general, $(\tau^{\mathrm{v}}(B)^*)_G \neq (\tau^{\mathrm{v}}(B)_G)^*$ (an example is given in Appendix A of Part II). However, if all finite-dimensional representations of G_b are decomposable (in particular, if G_b is compact), it is easy to prove that these two bundles coincide.

The set $C^\infty(\tau(B))_G$ of all G-invariant vector fields is in bijective correspondence with the set of all sections of the reduced bundle, $C^\infty(\tau(B)_G)$. But (5) implies that

$$\tau(B)_G \cong \tau^{\mathrm{v}}(B)_G \oplus \tau(N),$$

hence, there exits a bijective correspondence between the G-invariant vector fields and the couples of a "scalar field" (a section of $\tau^{\mathrm{v}}(B)_G$) and a vector field on N. The above construction, however, does not determine a fixed splitting of (5) because $\tau(N)$ is not canonically embedded in $\tau(B)_G$. One needs additional information to fix a certain splitting. The case of G-invariant differential forms is completely analogous.

Remark 2.3. If C is a G-invariant submanifold of B, the restriction to C yields naturally the SES

$$0 \to \tau(C) \xrightarrow{m} \tau(B)_C \xrightarrow{n} \nu(C) \to 0 \tag{6}$$

of vector bundles over C; here $\nu(C)$ is the quotient bundle, $\tau(B)_C/\tau(C)$. The SES (6) and its dual are G-intertwining, which implies the G-invariance of $m(\tau(C))$ and $n^*(\nu^*(C))$ as subbundles of $\tau(B)_C$ and $\tau^*(B)_C$, respectively.

Remark 2.4. Let $\zeta \in C^\infty(\tau(B))$ and $\psi \in C^\infty(\tau^*(B))$ be a G-invariant vector field and one-form, respectively. The local coordinates (x^μ) of B ($\mu = 1, \dots, \dim B$) induce local coordinates $(x^\mu, \mathrm{d}x^\mu)$ of $\tau(B)$ and $(x^\mu, \partial/\partial x^\mu)$ of $\tau^*(B)$. In these coordinates, the conditions for G-invariance of the sections of $\tau(B)$ and $\tau^*(B)$ read

$$g(\zeta)^\mu(b) := (t_{g*}\zeta)^\mu(b) = \frac{\partial t_g^\mu(t_{g^{-1}}(b))}{\partial x^\nu}\zeta^\nu(t_{g^{-1}}(b)) = \zeta^\mu(b),$$
$$g(\psi)_\mu(b) := (\hat{t}_g\psi)_\mu(b) = \frac{\partial t^\nu_{g^{-1}}(b)}{\partial x^\mu}\psi_\nu(t_{g^{-1}}(b)) = \psi_\mu(b),$$

where $g \in G$.

Example 2.1. Let us consider the example of the dimensional reduction of the $O_0(3)$-invariant vector fields on $\mathbb{R}^3$, i.e. let $B := \mathbb{R}^3\backslash\{\mathbf{0}\}$ and let $G := O_0(3)$ act by its tangent lift on $\tau(\mathbb{R}^3)$. (The subscript "0" means "the connected component of the unit element".) The orbits of the action of $O_0(3)$ on $\mathbb{R}^3$ are the spheres centered at the origin $\mathbf{0} := (0, 0, 0)$.

For a base N of the reduced bundle (2) we can choose any submanifold diffeomorphic to $B/G \cong \mathbb{R}_+$, e.g. $N := \{(\chi(z), 0, z)|z > 0\}$, where $\chi : \mathbb{R}_+ \to \mathbb{R}$ satisfies $\chi'(z) > 0$ for any $z > 0$.

For each $\mathbf{b} := (x, y, z) \in B$, the vertical subspace, $\tau^{\mathrm{v}}(B)_{\mathbf{b}}$, is the plane $\tau(S_b)_{\mathbf{b}}$ tangent to the sphere S_b of radius $b := \sqrt{x^2 + y^2 + z^2}$ at point $\mathbf{b}$. The one-dimensional quotient space $p^*\tau(B/G)_{\mathbf{b}} = \tau(B)_{\mathbf{b}}/\tau^{\mathrm{v}}(B)_{\mathbf{b}}$ can be realized as any subspace of $\tau(B)_{\mathbf{b}}$ transversal to $\tau^{\mathrm{v}}(B)_{\mathbf{b}}$.

The stationary subgroup $G_{\mathbf{b}}$ consists of the rotations around the straight line through the points $\mathbf{0}$ and $\mathbf{b}$. It acts freely on $\tau(S_b)_{\mathbf{b}}$, so st $\tau^{\mathrm{v}}(B)_{\mathbf{b}} =$ st $\tau(S_b)_{\mathbf{b}}$ consists only of the zero vector at $\mathbf{b}$. Thus, $(\tau^{\mathrm{v}}(B)_G)_{\mathbf{b}}$ consists only of the zero vector at $\mathbf{b}$, $(\tau(B)_G)_{\mathbf{b}}$ is the one-dimensional subspace of the radial vectors at $\mathbf{b}$, and $((p^*\tau(B/G))_G)_{\mathbf{b}}$ can be realized as $\tau(N)_{\mathbf{b}}$.

3. Differential operators on vector bundles

In this section we briefly introduce the necessary facts and notations concerning the theory of differential operators on vector bundles in terms of jets of sections. The interested reader can find a succinct introduction to the theory of jet bundles in [7]; detailed expositions (with numerous examples) can be found in [6,9], and more mathematical aspects in [1].

Let ξ be a vector bundle over V, $I_b(B)$ be the ideal of the ring $C^\infty(B)$ consisting of all functions vanishing at $b \in B$, and $I_b^k(B) \subset C^\infty(B)$ be the ideal of functions representable as a product of k functions from $I_b(B)$. Let $Z_b^k(\xi)$ stand for the vector space of those sections of ξ which are products of elements of $I_b^{k+1}(B)$ and $C^\infty(\xi)$, and $J^k(\xi)_b$ be the quotient space

$$J^k(\xi)_b := C^\infty(\xi)/Z_b^k(\xi).$$

The canonical linear mapping $C^\infty(\xi) \to J^k(\xi)_b : \psi \mapsto J^k(\psi)(b)$ maps a section ψ into its *k-jet* at $b \in B$, $J^k(\psi)(b)$. The k-jet is the coordinate-free concept for the section ("the field") ψ and its derivatives up to order k at b. Hence, $J^k(\psi)(b) = J^k(\phi)(b)$ means that in some (and hence in all) local coordinates, the Taylor series of ψ and ϕ around b agree up through order k. The mapping

$$J^k : C^\infty(\xi) \to C^\infty(J^k(\xi)) : \psi \mapsto J^k(\psi)$$

is called a *k-jet* of the section ψ. (To avoid confusion, we would like to alert the reader that we use the notation $J^k(\xi)$ both for the bundle of jets and for its total space; in the above formula, $C^\infty(J^k(\xi))$ stands for the set of all sections of the vector bundle $J^k(\xi)$, while in the formula below it means the total space.)

The vector bundle structure of the union

$$J^k(\xi) := \bigcup_{b\in B} J^k(\xi)_b$$

is natural: the local coordinates (x^μ, u^a) ($\mu = 1, \dots, \dim B$; $a = 1, \dots, \dim \xi$) of ξ generate local coordinates $(x^\mu, u^a, u^a_{\mu_1}, \dots, u^a_{\mu_1,\dots,\mu_k})$, $(1 \le \mu_1 \le \dots \le \mu_i \le \dim B; i = 1, \dots, k)$ of $J^k(\xi)$, where

$$u^a_{\mu_1,\dots,\mu_i}(J^k(\psi)(b)) := \frac{\partial^i}{\partial x^{\mu_1}, \dots, \partial x^{\mu_i}} \psi^a(b), \tag{7}$$

and the transition functions follow from the well-known formulae for transformation of partial derivatives under a change of variables. From the definition, $J^0(\xi) = \xi$. The dimension of the fibers of vector bundle $J^k(\xi)$ is

$$\dim J^k(\xi) = \binom{\dim B + k}{k} \dim \xi.$$

Let $\pi^{k,l} : J^k(\xi) \to J^l(\xi), k > l \geq 0$ denote the natural projections ("cutting off" all derivatives of order $l+1, \dots, k$).

Let ξ and η be vector bundles over B. A *differential operator* (DO) of order k from ξ to η is a mapping

$$D : C^\infty(\xi) \to C^\infty(\eta) : \psi \mapsto D\psi$$

such that $J^k(\psi)(b) = 0$ implies $D\psi(b) = 0$. If D is a linear mapping, it is called a *linear differential operator*. Let $\mathrm{Diff}_k(\xi, \eta)$ and $\mathrm{LDiff}_k(\xi, \eta)$ stand, respectively, for the vector spaces of DOs and linear DOs of order k from ξ to η.

There exists a bijective correspondence between the space $\mathrm{LDiff}_k(\xi, \eta)$ and the space $\mathrm{Hom}(J^k(\xi), \eta)$ of vector bundle morphisms over the identity in B. This correspondence is based on the fact that every vector bundle morphism $T : \zeta \to \eta$ over the identity in B generates naturally a mapping

$$T_* : C^\infty(\zeta) \to C^\infty(\eta) : \phi \mapsto T_*(\phi) := T \circ \phi,$$

and each linear DO $D \in \mathrm{LDiff}_k(\xi, \eta)$ corresponds to a vector bundle morphism $T \in \mathrm{Hom}(J^k(\xi), \eta)$ such that $D = T_* \circ J^k$, i.e. that the diagram

$$\begin{tikzcd} C^\infty(\xi) \arrow[r, "J^k"] \arrow[d, "D"'] & C^\infty(J^k(\xi)) \arrow[dl, "T_*"] \\ C^\infty(\eta) & \end{tikzcd}$$

commutes. The morphism T is called the *total symbol* of D. For a nonlinear DO this construction is analogous, but in this case T is simply a fiber preserving mapping.

Parenthetically, in the language of category theory (see, e.g. [3]), the couple $(J^k, J^k(\xi))$ is a universal element for the covariant functor $\mathcal{F} = (\mathcal{F}_{\mathrm{Ob}}, \mathcal{F}_{\mathrm{Mor}})$ from the category $\mathcal{V}(B)$ of vector bundles over B to the category of linear DOs of order k from $\xi \in \mathcal{V}(B)$ to another vector bundle over B. Namely, if $\eta, \zeta \in \mathcal{V}(B)$ and $T \in \mathrm{Hom}(\eta, \zeta)$, then

$$\mathcal{F}_{\mathrm{Ob}}(\eta) = \mathrm{LDiff}_k(\xi, \eta), \qquad \mathcal{F}_{\mathrm{Mor}}(T) = T_* = T \circ .$$

One can differentiate simultaneously both sides of the differential equation $D\psi = \phi$, thus obtaining a differential operator of higher order acting on ψ. The formal definition of this is the following. Let $D = T_* \circ J^k \in \mathrm{Diff}_k(\xi, \eta)$. There exists a unique fiber preserving mapping $P^l(T) : J^{k+l}(\xi) \to J^l(\eta)$ such that the diagram

$$\begin{tikzcd} C^\infty(J^{k+l}(\xi)) \arrow[r, "P^l(T)_*"] & C^\infty(J^l(\eta)) \\ C^\infty(\xi) \arrow[u, "J^{k+l}"] \arrow[r, "D"'] & C^\infty(\eta) \arrow[u, "J^l"'] \end{tikzcd}$$

commutes. The lth *prolongation* of D is by definition

$$P^l(D) := P^l(T)_* \circ J^{k+1} = J^l \circ T_* \circ J^k \in \mathrm{Diff}_{k+l}(\xi, J^l(\eta)).$$

In general, $R^{k,l} := \ker P^l(T)$ is a family of linear subspaces of the vector bundle $J^{k+l}(\xi)$. A DO $D \in \mathrm{LDiff}_k(\xi, \eta)$ said to be *formally integrable* if for each $l \geq 0$ the following conditions are satisfied:

(a) $R^{k,l}$ is a vector subbundle of $J^{k+l}(\xi)$;
(b) the natural projection $\pi^{k+l+1,k+l} : R^{k,l+1} \to R^{k,l}$ is an epimorphism.

For a formally integrable DO, the subbundle $R^{k,0} \subset J^k(\xi)$ is called its *equation*.

The meaning of the formal integrability of a DO is that for a formally integrable DO $D \in \mathrm{Diff}_k(\xi, \eta)$, by differentiating l times the equation $D\psi = 0$, one can never obtain a condition on the derivatives of ψ that contains no derivatives of order $k + l$ but only lower order derivatives (see the example below).

The formal integrability of a differential operator can be established by using methods of homological algebra (for references see, e.g. [1]). An example of a differential operator that is not formally integrable is the following.

Example 3.1. Let ξ and η be vector bundles over $\mathbb{R}^3$ with typical fibers $\mathbb{R}$ and $\mathbb{R}^2$, respectively, and

$$D := \begin{pmatrix} \partial_{zz} - y\partial_{xx} \\ \partial_{yy} \end{pmatrix} \in \mathrm{LDiff}_2(\xi, \eta),$$

where x, y and z are the coordinates in $\mathbb{R}^3$. Then $\pi^{3,2} : R^{2,1} \to R^{2,0}$ is an epimorphism, but $\pi^{4,3} : R^{2,2} \to R^{2,1}$ is not, because the second prolongation of D yields the condition $u_{xxy} = 0$ which was not present in $R^{2,1}$. In fact, there are infinitely many conditions of this kind that appear in the higher prolongations, and the general solution of $D\psi = 0$ contains only 12 parameters:

$$\begin{aligned}\psi(x, y, z) = {} & xy(\alpha_1 z^3 + 3\alpha_2 z^2 + \alpha_3 z + \alpha_4) + y(\alpha_5 z^3 + \alpha_6 z^2 + \alpha_7 z + \alpha_8) \\ & + (\alpha_1 x^3 + 3\alpha_5 x^2 + \alpha_9 x + \alpha_{10})z + \alpha_2 x^3 + \alpha_6 x^2 + \alpha_{11} x + \alpha_{12}.\end{aligned}$$

This example is studied in detail in [8, Introduction].

4. Restriction of a differential operator to a submanifold

The dimensional reduction of an invariant DO $D \in \mathrm{Diff}_k(\xi, \eta)_G$ is closely related to the problem of restricting D to N, where N is a submanifold of B. By "restricting D to N", we mean constructing a DO $D_N \in \mathrm{Diff}_k(\xi_N, \eta_N)$ from D by means of the natural embedding $i : N \hookrightarrow B$. This procedure is not naturally defined, so in this section we will explain how it can be performed.

To understand the problem, let us first consider it in local coordinates. Let $\dim B = n$, $\dim N = \nu$, and let the local coordinates $x^1, \ldots, x^n$ be such that

$$N = \{x^{\nu+1} = \cdots = x^n = 0\}.$$

We will call $x_1, \dots, x^\nu$ "internal for N", and $x^{\nu+1}, \dots, x^n$ "external for N" coordinates. Let $\psi \in C^\infty(\xi)$, and let $\psi \circ i \in C^\infty(\xi_N)$ be its restriction to N.

If we calculate the k-jet of ψ, and then restrict $J^k(\psi)$ to N, we obtain $J^k(\psi) \circ i \in C^\infty(J^k(\xi)_N)$. For each $b \in N$, $J^k(\psi)(b)$ contains derivatives with respect to all coordinates, $x^1, \dots, x^n$. At the same time, if we first restrict ψ to N and then take the k-jet of $\psi \circ i$, we obtain $J^k(\psi \circ i) \in C^\infty(J^k(\xi_N))$. Since $\psi \circ i$ depends only on the internal for N coordinates, $x^1, \dots, x^\nu$, its k-jet, $J^k(\psi \circ i)$, contains derivatives with respect to these coordinates only. Thus, the dimensions of $J^k(\xi)_N$ and $J^k(\xi_N)$ are

$$\binom{n+k}{k} \dim \xi \quad \text{and} \quad \binom{\nu+k}{k} \dim \xi,$$

respectively.

If D contains only differentiations with respect to internal for N coordinates (in which case we will say that D is *internal for N*), the restricted DO D_N is simply equal to D. What to do, however, if D is not internal for N?

Let us first note that there exists a natural projection $j^k : J^k(\xi)_N \to J^k(\xi_N)$ which simply is "cutting off" all non-internal derivatives, i.e. those containing at least one external for N partial derivative. The relationship between the total symbols of D and D_N can be simply expressed with the help of j^k by means of the diagram

$$\begin{tikzcd} C^\infty(J^k(\xi)_N) \arrow[r, "j^k_*"] \arrow[d, "(T_*)_N"'] & C^\infty(J^k(\xi_N)) \arrow[dl, "(T_N)_*"] \\ C^\infty(\eta_N) & \end{tikzcd}$$

Clearly, for an internal DO D, the total symbol of D_N is defined naturally by $T = T_N \circ j^k$. If the DO is not internal, one needs some additional information.

Let I^k_N be the subbundle of $J^k(\xi)_N$ consisting of the k-jets of all sections of ξ that vanish on N:

$$I^k_N := \{J^k(\psi)(b) | \psi \in C^\infty(\xi) \ \text{s.t.} \ \psi(b) = 0 \ \forall b \in N\}.$$

In other words, I^k_N consists of those elements of $J^k(\xi)_N$ all internal for N derivatives of which (including the zeroth derivatives) are 0. Using the notations (7), the coordinates of the elements of I^k_N satisfy

$$u^a = 0, \qquad u^a_{\mu_1,\dots,\mu_i} = 0, \qquad 1 \le \mu_1 \le \cdots \le \mu_i \le \nu, \ \ i = 1, \dots, k.$$

If $i^k : I^k_N \hookrightarrow J^k(\xi)_N$ is the natural embedding, then clearly $j^k \circ i^k = 0$, and the horizontal sequence in the diagram below is exact:

$$\begin{tikzcd} 0 \arrow[r] & I^k_N \arrow[r, "i^k"] & J^k(\xi)_N \arrow[r, "j^k"] \arrow[d, "T"'] & J^k(\xi_N) \arrow[r] \arrow[l, bend right, "S"'] \arrow[dl, "T_N"] & 0 . \\ & & \eta_N & & \end{tikzcd} \tag{8}$$

The essence of the problem of restricting a DO to N is that while $J^k(\xi)_N \cong I_N^k \oplus J^k(\xi_N)$, the bundle $J^k(\xi_N)$ is not naturally embedded in $J^k(\xi)_N$. Therefore, to define T_N, one needs to choose a fiber preserving mapping $S : J^k(\xi_N) \to J^k(\xi)_N$ over the identity in N such that

$$j^k \circ S = \text{identity in } J^k(\xi_N). \tag{9}$$

Then the total symbol of D_N is defined by

$$T_N = T \circ S. \tag{10}$$

The condition (9) on S guarantees that for an internal for N DO, (10) yields a DO coinciding with the natural restriction of D to N. The mapping S is called a *splitting relation*.

One can define S by specifying its image $S(J^k(\xi_N)) \subset J^k(\xi)_N$, which can be defined as a kernel of the total symbol $\mathcal{T}$ of some DO $\mathcal{D}$ of order k acting on $C^\infty(\xi)$.

To clarify the matters, let us consider the following example (more examples will be given in Part II).

Example 4.1. Let $B := \mathbb{R}^2$ and (x, y) be the Cartesian coordinates, let ξ be a globally trivial bundle over B with fiber $\mathbb{R}$, and let $N := \{y = 0\} \subset B$.

Then the fiber coordinates of $J^2(\xi)_N$ are $(u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})$, while the fiber coordinates of $J^2(\xi_N)$ are only the ones that do not contain y-derivatives, i.e. (u, u_x, u_{xx}).

The Laplacian $\Delta := \partial_{xx} + \partial_{yy} \in \mathrm{LDiff}_2(\xi, \xi)$ is not internal for N because it contains y-derivatives.

Let us choose the splitting condition S to be defined by specifying its image, $S(J^2(\xi_N)) := \ker \mathcal{T}$, where $\mathcal{T}$ is the total symbol of the DO $\mathcal{D} := \partial_{xx} - \partial_{yy} + 3\partial_x$. Then u_{yy} can be expressed as $u_{yy} = u_{xx} + 3u_x$, so the restricted to N Laplacian becomes $\Delta_N = 2\partial_{xx} + 3\partial_x \in \mathrm{LDiff}_2(\xi_N, \xi_N)$.

Remark 4.1. In general, the order of D_N can be higher than the order of D; for example, if in the above example we had chosen $S = u_{xxx} - 2u_{xx} - u_{yy} - 2u_x$, then the reduced Laplacian would have become $\Delta_N = \partial_{xxx} - \partial_{xx} - 2\partial_x \in \mathrm{LDiff}_3(\xi_N, \xi_N)$. This can happen when S contains derivatives of order higher than the order of D. However, a DO $D \in \mathrm{Diff}_m(\xi, \eta)$ can be considered as an element of $\mathrm{Diff}_l(\xi, \eta)$ for all $l > m$. Therefore, all constructions from this section hold if by k we mean the maximum of the order of D and the order of the highest derivative in the splitting relation S.

5. Dimensional reduction of invariant differential operators

Let ξ and η be G-reducible vector bundles over the manifold B with the same action t of G on B. Then the actions of $g \in G$ on $C^\infty(\xi)$ and $C^\infty(\eta)$,

$$g^\xi : \psi \mapsto T_g^\xi \circ \psi \circ t_g^{-1}, \qquad g^\eta : \chi \mapsto T_g^n \circ \chi \circ t_g^{-1},$$

generate an action of g on the DOs:

$$g(D) := g^{\eta} \circ D \circ (g^{\xi})^{-1}, \quad D \in \mathrm{Diff}_k(\xi, \eta).$$

A DO $D \in \mathrm{Diff}_k(\xi, \eta)$ is *G-invariant* if $g(D) = D$. The space or all G-invariant DOs will be denoted by $\mathrm{Diff}_k(\xi, \eta)_G$. If $\psi \in C^{\infty}(\xi)_G$ and $D \in \mathrm{Diff}_k(\xi, \eta)_G$, then $D\psi \in C^{\infty}(\eta)_G$, therefore each G-invariant DO D generates a reduced DO

$$D_G := \theta^{-1} \circ D \circ \theta : C^{\infty}(\xi_G) \to C^{\infty}(\eta_G),$$

where θ is the map defined in (1). To construct D_G, one must do the following:

(a) find the stationary subbundles $\mathrm{st}\,\xi$ and $\mathrm{st}\,\eta$;
(b) choose a submanifold $N \cong B/G$ transversal to the G-orbits in B;
(c) restrict the DO $D : C^{\infty}(\mathrm{st}\,\xi) \to C^{\infty}(\mathrm{st}\,\eta)$ to N.

As pointed out in Section 4, to perform step (c) in this algorithm, one needs a fiber preserving mapping $S : J^k(\xi_N) \to J^k(\xi)_N$. In the case of dimensional reduction of a G-invariant DO, there exists a natural splitting of the SES (8) due to the possibility to extend uniquely by G-invariance each section of $\mathrm{st}\,\xi_N$ to a section over a neighborhood of N. In the rest of this section, we explain how this can be done.

The *Lie derivative* $L : C^{\infty}(\xi) \to C^{\infty}(\mathfrak{g}^* \otimes \xi)$ of the action (T, t) of the Lie group G on the section $\psi \in C^{\infty}(\xi)$ is defined by

$$L\psi(Y) := \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{tY}(\psi)|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} T_{\exp(tY)} \circ \psi \circ t_{\exp(-tY)}|_{t=0},$$

where $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g}$ of G, and $Y \in \mathfrak{g}$. The stationary subbundle $\mathrm{st}\,\xi$ is invariant under the action of G on ξ, hence the Lie derivative has a natural restriction to $\mathrm{st}\,\xi$ which will also be denoted by L.

Let

$$\tilde{L} : J^1(\mathrm{st}\,\xi) \to \mathfrak{g}^* \otimes \mathrm{st}\,\xi$$

be the total symbol of the Lie derivative $L \in \mathrm{LDiff}_1(\mathrm{st}\,\xi, \mathfrak{g}^* \otimes \mathrm{st}\,\xi)$. The $(k-1)$th prolongation of L provides a splitting relation for each G-invariant DO $D \in \mathrm{Diff}_k(\xi, \eta)_G$. Indeed, let

$$P^{k-1}(\tilde{L}) : J^k(\mathrm{st}\,\xi) \to J^{k-1}(\mathfrak{g}^* \otimes \mathrm{st}\,\xi)$$

be the symbol of the $(k-1)$th prolongation of L,

$$R_N^k := (\ker P^{k-1}(\tilde{L}))_N \subset J^k(\mathrm{st}\,\xi)_N$$

be the restriction of its kernel to N, and i^k and I_N^k have the same meaning as in (8) (with $\mathrm{st}\,\xi$ instead of ξ). Then it can be shown that R_N^k defines a splitting of (8), i.e. it is a complementary subbundle of $i^k(I_N^k)$ in $J^k(\mathrm{st}\,\xi)_N$:

$$J^k(\mathrm{st}\,\xi)_N = i^k(I_N^k) \oplus R_N^k.$$

This means that for each $b \in N$, all non-internal for N derivatives in $J^k(\psi)(b)$ can be expressed in terms of the purely internal derivatives by solving the system

$$P^{k-1}(\tilde{L})_N = 0,$$

which provides us with an algorithm for restriction to N, and hence for dimensional reduction of each G-invariant DO of order k.

A very important for this algorithm fact is that the Lie derivative is a formally integrable differential operator. Due to the formal integrability of L, in order to reduce a G-invariant DO of order k, one does not need to consider $P^l(\tilde{L})_N = 0$ for $l > k-1$, so the above theory always yields an algorithm consisting of finitely many steps. This, in particular, implies that in the process of reduction of a G-invariant DO the order of the operator does not increase, i.e. if $D \in \mathrm{Diff}_k(\xi, \eta)_G$, then $D_G \in \mathrm{Diff}_m(\xi_G, \eta_G)$, where $m \le k$.

Remark 5.1. The short exact sequence of vector bundles in the diagram (8) admits different splittings. However, in the procedure of reduction of an *invariant* DO, it is the requirement for *G-invariance of the sections* of ξ and η that yields a unique splitting. This splitting determines the reduced DO.

In the proposed procedure for reduction, one needs only to calculate the prolongation of the Lie derivatives and to solve *algebraic* equations. This is generally easier than finding an ansatz for the invariant sections (to find such an ansatz, one would have to solve a system of first order partial differential equations).

In this "local" approach to dimensional reduction, one does not need to know a global ansatz for the invariant sections, but only their Taylor expansion up to order k around N (for DOs of order k).

Below we illustrate the method of dimensional reduction of invariant differential operators on elementary examples. Notice how simple the calculations are (as opposed to, say, change of variables), and the fact that, although the reducing group in the second example is non-compact, the reduction procedure is essentially the same.

Example 5.1. Let us consider the $O_0(3)$-reduction to the Laplace operator acting on scalar functions in $\mathbb{R}^3$.

For a base of the reduced bundle let us choose the transversal to the orbits of $O_0(3)$ manifold $N := \{(0, 0, z) | z > 0\}$.

The generators of $O_0(3)$ are

$$X_1 := x\partial_y - y\partial_x, \qquad X_2 := x\partial_z - z\partial x, \qquad X_3 := y\partial_z - z\partial_y.$$

The infinitesimal symmetry condition $X_2 f = 0$ implies $\partial_x f = (x/z)\partial_z f$. The restriction to the submanifold N of the x-component of the first prolongation of this condition yields $\partial_{xx} f = (1/z)\partial_z f$ on N. Similarly, the condition $X_3 f = 0$ gives $\partial_{yy} f = (1/z)\partial_z f$ on N. Plugging these equations in $\Delta := \partial_{xx} + \partial_{yy} + \partial_{zz}$, we obtain the reduced Laplace operator,

$$\Delta_{O_0(3)} = \frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{2}{z}\frac{\mathrm{d}}{\mathrm{d}z},$$

acting on functions defined on N—as expected, for this choice of N, $\Delta_{O_0(3)}$ is the radial part of the Laplacian.

Example 5.2. As an example of reduction with a non-compact reducing group consider the $O_0(1,2)$-reduction of the $(1+2)$-dimensional D'Alembertian

$$\Box_3 := \partial_{xx} + \partial_{yy} - \partial_{zz}$$

in the interior of the "future" light cone. Let $N := \{(0,0,z)|z > 0\}$.

The generators of $O_0(1,2)$ are $x\partial_y - y\partial_x$, $x\partial_z + z\partial_x$, and $y\partial_z + z\partial_y$. The same calculation as in the previous example yields $\partial_{xx} f = -(1/z)\partial_z f$, $\partial_{yy} f = -(1/x)\partial_z f$ on N, so the reduced wave operator is

$$(\Box_3)_{O_0(1,2)} = -\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{2}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right).$$

Note that the $O_0(3)$-invariant harmonic functions satisfy the same equation as the $O_0(1,2)$-invariant solutions of the wave equation.

6. Dimensional reduction of a group action

We need some details concerning the dimensional reduction of a group action. Let $\xi = (E,\pi,B)$ be a G-reducible vector bundle, the Lie group K act on ξ by vector bundle morphisms (F,f), and the actions of G and K commute. Due to the mutual commutativity of the actions (T,t) and (F,f) of G and K on ξ, the action of $k \in K$ maps the G-invariant sections of ξ into G-invariant ones (i.e. $C^\infty(\xi)_G$ is a K-invariant subset of $C^\infty(\xi)$). This fact allows us to define a natural action (F_G, f_G) of K on the reduced bundle $\xi_G = (\mathrm{st}\,\xi_N, \pi_G, N)$ as follows. Let $k \in K$, $b \in N$, $\sigma \in (\xi_G)_b = \mathrm{st}\,\xi_b$, and let $g \in G$ be such that $t_g \circ f_k(b) \in N$, then

$$(t_G)_k(b) := t_g \circ f_k(b), \qquad (F_G)_k(\sigma) := T_g \circ F_k(\sigma) \in (\xi_G)_{(t_G)_k(b)}.$$

The action k_G of $k \in K$ on $C^\infty(\xi_G)$ is

$$k_G : C^\infty(\xi_G) \to C^\infty(\xi_G) : \gamma \mapsto k_G(S) := \theta^{-1} \circ k \circ \theta \circ S.$$

Induced representations are an example of dimensional reduction of a group action.

Let ξ and η be G-reducible vector bundles over B with the same action t of G on B. Let the Lie group K act on ξ and η by vector bundle morphisms with the same action f of K on B. Let the action (T^ξ, t) of G on ξ commute with the action (F^ξ, f) of K on ξ, and the same hold for the actions (T^η, t) and (F^η, f) on η. It is easy to see that if a DO $D \in \mathrm{Diff}_k(\xi,\eta)$ is simultaneously K- and G-invariant, then the reduced DO $D_G \in \mathrm{Diff}_k(\xi_G, \eta_G)$ is invariant under the reduced actions (F_G^ξ, f_G) and (F_G^η, f_G) of K on ξ_G and η_G.

Acknowledgements

We would like to express our gratitude to Lubomir Chorbadjiev for the innumerable discussions on symmetry and reduction. We are indebted to Prof. Luis J. Boya who read the article and made many valuable comments. We are grateful to Prof. Rafael de la Llave for his help and encouragement. N.P.P. was partially supported by Grants DMS 9802156 and TARP-003658-071. We would also like to thank the referee whose comments helped us clarify some important points.

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